MKD space

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Intro to Morrison–Kawamata dream space.
(MKD space)
(joint work with Xingying Li
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/$\mathbb{C}$ X, Y = normal projective.

X = Fano type. if $\exists \Delta \geq 0$ st (X, Δ) is klt
$-(K+\Delta)$ is ample

(eg toric variety $\rightarrow$ D-MMP $\forall D$ works)

Question : Any D-MMP works on FT variety?
(before [BCHM]

Def Mori dream space := $\underline{\text{MDS}}$

X = MDS if

① X = $\mathbb{Q}$-factorial, $\mathrm{Pic}(X)_{\mathbb{Q}} = N^1(X)_{\mathbb{Q}}$

② $\mathrm{Nef}(X) = \sum_{i=1 \sim n} \mathbb{R}_{\geq 0} D_i$ D_i = semiample.

③ $\exists$ only finitely many SQM $f_i : X \dashrightarrow X_i$ st $i = 1 \sim m$
each X_i satisfies ①, ② &
$\overline{\mathrm{Mov}(X)} = \bigcup_{i=1 \sim m} f_i^* \mathrm{Nef}(X_i)$.

Well known.

- X = MDS iff $\mathrm{Cox}(X)$ is f.g.
- D-MMP for any D works on MDS
- [BCHM] FT are MDS.

X = $\underline{\text{Calabi–Yau}}$ type if $\exists \Delta \geq 0$ (X, Δ) klt
CYT. $K + \Delta \sim_{\mathbb{R}} 0$

FT $\subseteq$ CYT

What is expected for CXT.

1) D-MMP for eff D w/ good minimal models.

(known in $\dim \le 3$)

(D-MMP for non eff D is unknown).

$$\left(\begin{array}{l} D\text{-MMP} = \varepsilon D\text{-MMP} \quad 0 < \varepsilon \ll 1 \\ \qquad\quad\; = (K_X + \Delta + \varepsilon D)\text{-MMP}. \\ \qquad\qquad \text{klt} \quad \text{lc} \quad X \end{array}\right)$$

2) Morrison–Kawamata cone conjecture.

(X, Δ) klt CY pair.

psAut(X,Δ) acts on $\overline{\mathrm{Mov}}^e(X)$

$\shortparallel$ small birational self map of X. $\qquad$ $\overline{\mathrm{Mov}}(X) \cap \mathrm{Eff}(X)$

Aut(X,Δ) acts on $\mathrm{Nef}^e(X)$

$\shortparallel$ automorphism of X preserving Supp Δ $\qquad$ $\mathrm{Nef}(X) \cap \mathrm{Eff}(X)$

① $\exists$ fundamental domain $\Pi \subseteq \overline{\mathrm{Mov}}^e(X)$

rat'l $\updownarrow$ polyhedral. for the action psAut(X,Δ)

② $\exists$ fundamental domain $\Pi \subseteq \mathrm{Nef}^e(X)$ for Aut(X,Δ).

Π = fundamental domain of C. by Γ.

$\Rightarrow \quad \Gamma \cdot \Pi = C$

$\sigma \Pi = \Pi$ only if $\sigma = \mathrm{id}$.

otherwise $\sigma \Pi \cap \mathrm{Int}\, \Pi = \phi$.

(known in $\dim \le 2$.

partially known in ≥ 3)

(known in dim ≤ 2.
partially known in ≥ 3)

$$\mathrm{FT} \overset{\forall D\text{-MMP}}{\subseteq} \mathrm{MDS}$$

$$\cap \qquad\qquad \cap$$

$$\mathrm{CYT} \subseteq \boxed{\;??\;} \leftarrow$$

$\uparrow$ D-MMP w/ $P \geq 0$
MK cone conj

Def X = Morrison–Kawamata dream space
(MKD space)

if
① X = $\mathbb{Q}$-factorial.
② any eff $\mathbb{R}$-Cartier D has good min model.
③ $\exists$ rational polyhedral cone $\Pi \subseteq \mathrm{Mov}(X)$
st $\mathrm{PsAut}(X) \cdot \Pi = \mathrm{Mov}(X)$.

→ ④ $\mathrm{Eff}(X)$ satisfies "local factoriality of cn models".

Def. Any set $S \subseteq \mathrm{CDiv}(X)$ satisfies local fact of cn models if
$\forall D \in S \quad \forall P = \mathrm{conv}(E_i \mid i = 1 \sim n\} \ni D$
(E_i: eff $\mathbb{Q}$-Cartier.)

$\exists$ open neighborhood $\mathcal{U}$ of D in P st

$$\begin{tikzcd} X \arrow[r, "f_B"] \arrow[d, "f_D"'] & X_B \arrow[dl, "h"] \\ X_D & \end{tikzcd}$$

$\forall B \in \mathcal{U}$ $\exists h : X_B \to X_D$ st $f_B \circ h = f_D$

plays the role of "cone theorem."

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Proposition. The following S satisfies LF of cn models

① $\mathrm{Nef}(X) \supseteq S$ = rat'l polyhedral cone.

② (Assume that any eff $\mathbb{R}$-Cartier div have good min models.)

1) $S = \mathrm{cone} \sum \mathbb{R}_{\geq 0} D_i$ s.t. $R(X, D_1, \dots D_m)$ is f.g.

D_i = eff $\mathbb{Q}$-Cartier div, $i = 1 \sim m$

2) any set $S' \subseteq \mathrm{CDiv}(X)$ where $X = CY$.

(S = LF cn model, $f: X \dashrightarrow Y$ SQM, $f_* S$ satisfies LF cn models)

(X, Δ) klt CY pair.

Take $D \in S$ $\quad$ $\forall D \in P = \mathrm{conv}(E_i)$ $\quad (0 < \varepsilon \ll 1)$

$\exists$ cn model for E_i = cn model for εE_i

= cn model for $(X, \Delta + \varepsilon E_i)$

By 1) $R(X, K + \Delta + \varepsilon E_1, \dots K + \Delta + \varepsilon E_n)$ is f.g. by DHP.

$= R(X, E_1 \dots E_m)$

$\Rightarrow$ cone $(E_1 \dots E_m) \to$ LF cn

so does the set S. $\square$

系 Corollary.

1) MDS $\subseteq$ MKD space

2) CYT ($\Leftarrow$ $\exists$ good min models + MK cone conj) $\subseteq$ MKD space.

2 ~~① ⊕~~ –

(MKD $\times$ MKD $=$ MKD)

③ X = MDS

$\overline{\mathrm{Mov}}(X)$ is decomposed by SQM.

$\Pi = \overline{\mathrm{Mov}}(X)$

④ $\mathrm{Eff}(X)$ = spanned by finitely many eff div. $\Rightarrow$ LF on models

Theorem (MKD space looks locally like MDS within Π.)

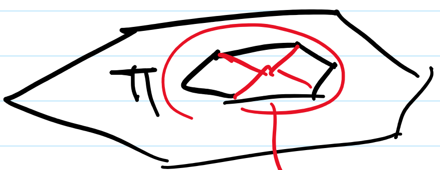
X = MKD iff

① X = $\mathbb{Q}$ factorial

② $\exists$ rat'l polyhedral cone $\Pi \subseteq \overline{\mathrm{Mov}}^e(X)$ st $\mathrm{PsAut}(X)\,\Pi = \overline{\mathrm{Mov}}^e(X)$

③ $\exists$ finitely many SQM $f_i : X \dashrightarrow X_i$ st $\Pi \subseteq \bigcup f_i^* \mathrm{Nef}(X_i)$ and $\underline{\Pi \cap f_i^* \mathrm{Nef}(X_i)}$ is rat'l polyhedral

④ $f_{i*}D$ is semiample for each eff $D \in \Pi \cap f_i^* \mathrm{Nef}(X_i)$



looks like the Mov of MDS.

Question : characterize ~~the~~ MKD space

Question : Characterize ~~the~~ MKD space by Cox(X) ??

proposition : X = MKD space
$f : X \dashrightarrow Y$ = $\mathbb{Q}$-factorial bir contraction
$\Rightarrow$ Y is also MKD space.

Question : If f = fibration, then Y = MKD space
(true for CY T, MDS)

Theorem : X = MKD space. CY T $\subseteq$ MKD

① $\forall D$ = eff. D-MMP w/ scaling of an ample divisor works & terminate with good min models
(each intermediate models are MKD spaces)

② MK cone conjecture holds for X.

$\overline{\mathrm{Mov}}^e(X) = \overline{\mathrm{Mov}}(X)$ ($\because \exists$ good min models for $D \geq 0$)

$\hookrightarrow$ Π = fundamental domain for MK cone conjecture.

$\mathrm{Nef}^e(X) \Rightarrow$ apply "Shokurov's polytope" & "Looijenga's results".

MDS

CYT

MKD